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ON DYNAMICAL SYSTEMS CLOSE TO A PRODUCT OF m ROTATIONS

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Abstract. We consider analytic one parameter families of vector fields and diffeomorphisms, including for a parameter value, the product of rotations in $\mathbb{R}^{2m} \times \mathbb{R}^n$ such that for positive values of the parameter the origin is a hyperbolic point of saddle type. We address the question of determining the limit stable invariant manifold when ε goes to zero as a subcenter invariant manifold when $\varepsilon = 0$.

1. **Introduction.** This paper is the natural continuation of some previous investigations [3, 4] about of the limit behaviour of invariant manifolds of hyperbolic points for families of vector fields of the form

$$X_\varepsilon(x) = Lx + \varepsilon g(x, \varepsilon)$$

in $\mathbb{R}^{2m} \times \mathbb{R}^n$ depending on the parameter $\varepsilon \geq 0$, with

$$L = \begin{bmatrix} 0 & -\beta_1 & & & & \\ \beta_1 & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & -\beta_m & \\ & & & \beta_m & 0 & \\ & & & & & \mathbf{0}_n \end{bmatrix}, \tag{1.1}$$

where $\mathbf{0}_n$ stands for the zero matrix in dimension n, β_k are non–zero real numbers and g is some regular function vanishing at the origin. We also deal with the case of diffeomorphisms

$$F_\varepsilon(x) = Ax + \varepsilon f(x, \varepsilon)$$

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with

$$A = \begin{bmatrix} G_1 & & & \\ & \ddots & & \\ & & G_m & \\ & & & \mathrm{Id}_n \end{bmatrix},$$

where G_j are rotations in $\mathbb{R}^2$. In [3] it has been considered the case of diffeomorphisms, assuming conditions on the $O(\varepsilon)$ terms of F_ε that ensure the existence for $\varepsilon > 0$ of an unstable manifold close to $\{0\} \times \mathbb{R}^n$ and a stable manifold close to $\mathbb{R}^{2m} \times \{0\}$. Assuming that $F_\varepsilon \in C^r$, it has been proved that the unstable manifold of F_ε goes to $\{0\} \times \mathbb{R}^n$ in the C^r topology.

In [4] the authors studied the stable manifold case, under analogous conditions, both for maps and vector fields, but restricting to the case $m = 1$. The limit manifold is obtained as the stable manifold of an auxiliary system independent of the parameter.

Here we consider the more general case in which the stable manifold of the origin is close to $E_1 \times E_2$ where E_1 and E_2 are suitable subspaces of $\mathbb{R}^{2m}$ and $\mathbb{R}^n$ respectively, for $\varepsilon > 0$ small. We restrict ourselves to the analytical case, hence to obtain a convergent normal form we need some arithmetical condition for the betas.

Families of this form appear as unfoldings of singularities. In particular the Hopf zero singularity, whose linear part is

$$\begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and the Hopf Hopf singularity, whose linear part is

$$\begin{bmatrix} 0 & -\omega_1 & & \\ \omega_1 & 0 & & \\ & & 0 & -\omega_2 \\ & & \omega_2 & 0 \end{bmatrix}.$$

The codimension two bifurcations associated to these singularities has attracted the attention of several authors, see [13, 9, 5, 7, 8, 10, 2]. Particularly Guckenheimer found that in the unfolding of the Hopf zero singularity there occurs the Shil'nikov bifurcation. Our results provide a tool to control, the invariant manifolds uniformly with respect to the small parameter ε.

The paper is organized as follows. In Section 2 we introduce the setting for the vector fields case and we state the main result, then in Section 3 we present some preliminary results and we construct the function which determines the limit invariant manifold. Finally in Section 4 we conclude the proof of our main theorem after having introduced and studied a normal form. Section 5 is devoted to the presentation of an analogous result in the diffeomorphisms setting.

2. **Setting and main result for vector fields.**

We are interested in vector fields whose corresponding differential equation have the form

$$\vec{x}' = L\vec{x} + \varepsilon g(\vec{x}, \varepsilon), \tag{2.1}$$

where L have been defined in (1.1).

Reordering the variables and rewriting $\vec{x} = (x, y, z) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^n$ we can express (2.1) in the form

$$X_\varepsilon : \begin{cases} x' = -B.y + \varepsilon g_1(x, y, z, \varepsilon), \\ y' = B.x + \varepsilon g_2(x, y, z, \varepsilon), \\ z' = \varepsilon g_3(x, y, z, \varepsilon) \end{cases} \tag{2.2}$$

with $B = \text{diagonal}\,[\beta_1, \ldots, \beta_m]$. We adopt here the notation such that diagonal $[C_1, \ldots, C_m]$ means a block matrix whose diagonal blocks are C_j and the off-diagonal blocks are zero; some or all blocks may be one dimensional.

In the following we shall assume that $g = (g_1, g_2, g_3)$ verifies $g(0, \varepsilon) = 0$ and that it is analytic. We shall have to take into account the linear part of g, so let us introduce here a notation used in the rest of paper: we develop g with respect to x, y, z and ε at 0:

$$g(x, y, z, \varepsilon) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + O(\varepsilon) + O(|(x, y, z)|^2). \tag{2.3}$$

To rewrite (2.2) in a more convenient form for successive computations we introduce complex coordinates:

$$x_j = x^j + iy^j \;\in \mathbb{C}, \qquad 1 \le j \le m, \tag{2.4}$$

and to shorten the notations, also a compact form:

$$x = (x_1, \ldots, x_m) \in \mathbb{C}^m, \qquad \bar{x} = (\bar{x}_1, \ldots, \bar{x}_m) \in \mathbb{C}^m. \tag{2.5}$$

Using the new variables $(x, \bar{x}, z)$, equation (2.2) can be rewritten as:

$$\left\{ \begin{array}{l} x' = iB.x + \varepsilon R(x, \bar{x}, z) \\ \bar{x}' = -iB.\bar{x} + \varepsilon \bar{R}(x, \bar{x}, z) \\ z' = \varepsilon S(x, \bar{x}, z) \end{array} \right\rangle + O(\varepsilon^2), \tag{2.6}$$

where $R(x, \bar{x}, z) = b_{11}.x + b_{12}.\bar{x} + b_{13}.z + O(|(x, \bar{x}, z)|^2)$ and $S(x, \bar{x}, z) = b_{31}.x + \bar{b}_{31}.\bar{x} + a_{33}.z + O(|(x, \bar{x}, z)|^2)$. The new coefficients b_{ij} here introduced, are related to the previous ones, a_{ij}, by:

$$b_{11} = (a_{11} + a_{22} + i(a_{21} - a_{12}))/2\,, \quad b_{12} = (a_{11} - a_{22} + i(a_{21} + a_{12}))/2, \tag{2.7}$$

$$b_{13} = a_{13} + ia_{23} \quad \text{and} \quad b_{31} = (a_{31} - ia_{32})/2\,. \tag{2.8}$$

Let us denote the diagonal elements of b_{11} by $b_{11}^1, \ldots, b_{11}^m$ and for further use let us introduce $R = (R^1, \ldots, R^m)$. We can now introduce two *generic hypotheses* on the hyperbolicity of the system.

Hypothesis 1. *The real parts of b_{11}^j, $j = 1, \ldots, m$, are different from zero. Hence, up to a permutation of the variables $x_1, \ldots, x_m$, we may, and will, assume that there is an index j_s with $0 \le j_s \le m$ such that*

$$\operatorname{Re} b_{11}^1 \le \cdots \le \operatorname{Re} b_{11}^{j_s} < 0 < \operatorname{Re} b_{11}^{j_s+1} \le \cdots \le \operatorname{Re} b_{11}^m. \tag{2.9}$$

(We make the convention that if $j_s = 0$ all b_{11}^j have positive real part and if $j_s = m$ all b_{11}^j have negative real part.) We can use this index to make the following splitting $x = (x^s, x^u)$ with $x^s = (x_1, \ldots, x_{j_s})$ and $x^u = (x_{j_s+1}, \ldots, x_m)$.

Hypothesis 2. *The matrix a_{33} is hyperbolic in the sense that all of its eigenvalues have real part different from zero. Hence, up to a linear change of variables only* *in $z \in \mathbb{R}^n$, we may, and will assume that there is a decomposition of $\mathbb{R}^n$ such that, with respect to it, a_{33} has the form*

$$a_{33} = \begin{bmatrix} a_{33}^s & 0 \\ 0 & a_{33}^u \end{bmatrix}, \tag{2.10}$$

where the eigenvalues of a_{33}^s, resp. a_{33}^u, have negative, resp. positive real part. Once again we use this decomposition to rewrite the variable $z \in \mathbb{R}^n = H_1 \oplus H_2$ as $z = (z^s, z^u)$. Let us set $k_s = \dim H_1$.

As already remarked to work in the analytical setting we need some arithmetical condition on the betas, to ensure convergence of formal objects; the hypothesis introduced here is weaker than the one previously used [4].

Hypothesis 3. *Let $\beta = (\beta_1, \ldots, \beta_m) \in \mathbb{R}^m$, then for $\vec{n} \in \mathbb{Z}^m$ we assume*

$$< \beta, \vec{n} >= 0 \quad \iff \quad \vec{n} = 0 \quad \textit{(non–resonance condition)},$$

and also

$$\sigma := \overline{\lim_{|\vec{n}| \to +\infty}} \frac{1}{|\vec{n}|} \log |\langle \beta, \vec{n} \rangle|^{-1} < \infty \quad \textit{(small divisors growth)}. \tag{2.11}$$

Remark 2.1. *As a consequence of this hypothesis $\beta_k \neq -\beta_l$ for every k, l (in particular $\beta_k \neq 0$) and $\beta_k \neq \beta_l$ for $k \neq l$.*

A widely used arithmetical condition in normal form problems is the Diophantine one, it is thus natural to compare our condition with this one.

Remark 2.2. *One says that β is Diophantine if there exist $K > 0$ and $\tau > 0$ such that*

$$|\langle \beta, \vec{n} \rangle| \geq K |\vec{n}|^{-\tau}, \qquad \forall \vec{n} \in \mathbb{Z}^m \setminus \{0\}. \tag{2.12}$$

It is clear that all Diophantine β, whatever are τ and K, do satisfy Hypothesis 3 with $\sigma = 0$ since

$$\overline{\lim_{|\vec{n}| \to +\infty}} \frac{\log |\langle \beta, \vec{n} \rangle|^{-1}}{|\vec{n}|} \leq \overline{\lim_{|\vec{n}| \to +\infty}} \frac{1}{|\vec{n}|} (\tau \log |\vec{n}| - \log K) = 0. \tag{2.13}$$

Hypothesis 3 tell us that the inverse of the small divisor can grow as an exponential (in the size of the integer component vector) instead of a power in case of a Diophantine β. Since the set of betas satisfying (2.12) *with $\tau > m$ has full Lebesgue measure* [1, 6] *then the set of betas satisfying* (2.11) *also has full Lebesgue measure in $\mathbb{R}^m$.*

Let us consider polar coordinates

$$x_j = r_j e^{i\theta_j}, \qquad \bar{x}_j = r_j e^{-i\theta_j}, \qquad 1 \leq j \leq m. \tag{2.14}$$

We write $r = (r_1, \ldots, r_m)$, $\theta = (\theta_1, \ldots, \theta_m)$, $r^s = (r_1, \ldots, r_{j_s})$, $r^u = (r_{j_s+1}, \ldots, r_m)$. Also $re^{i\theta} = (r_1 e^{i\theta_1}, \ldots, r_m e^{i\theta_m})$. For $k = 1, \ldots, m$ we define

$$C_1^k(r, z) = \frac{1}{(2\pi)^m} \int_{\mathbb{T}^m} \mathrm{Re}(R^k(re^{i\theta}, re^{-i\theta}, z) e^{-i\theta_k})\, d\theta \tag{2.15}$$

and

$$C_3(r, z) = \frac{1}{(2\pi)^m} \int_{\mathbb{T}^m} S(re^{i\theta}, re^{-i\theta}, z)\, d\theta. \tag{2.16}$$

Let Z^0 be the auxiliary vector field

$$Z^0 : \begin{cases} (r^s)' = C_1^s(r^s, 0, z), \\ (z^s)' = C_3^s(r^s, 0, z), \\ (z^u)' = C_3^u(r^s, 0, z), \end{cases} \tag{2.17}$$

where $C_1^s = (C^1, \dots, C^{j_s})$ and $C_3 = (C_3^s, C_3^u)$ according to the decomposition $z = (z^s, z^u)$.

In Proposition 3.6 we will see that the origin is a hyperbolic equilibrium point of Z^0 and its local stable manifold can be represented as the graph of

$$z^u = h^s(r^s, z^s). \tag{2.18}$$

Let $P_\rho \subset \mathbb{C}^p$ denote the polydisk $\{y \in \mathbb{C}^p \mid |y_i| < \rho\}$ and $B_\rho \subset \mathbb{C}^q$ the ball $\{z \in \mathbb{C}^q \mid \|z\| < \rho\}$, then we can state our main result:

Theorem 2.3. *Let X_ε be as in (2.1) or* (2.2) *and suppose that the functions g_j are analytic on $P_{\rho_0} \times B_{\rho_0} \subset \mathbb{C}^{2m} \times \mathbb{C}^n$, for some $\rho_0 > 0$. Assume that Hypotheses 1, 2 and 3 are satisfied. Consider the following local graph of the analytic function $(x^u, y^u, z^u) = h(x^s, y^s, z^s)$ defined by*

$$\{x^s, y^s, 0, 0, z^s, h^s(|x_1 + iy_1|, \dots, |x_{j_s} + iy_{j_s}|, z^s)\}, \tag{2.19}$$

where h^s is the function introduced in (2.18) *defined on $P_{\rho_1} \times B_{\rho_1} \subset \mathbb{C}^{j_s} \times \mathbb{C}^{k_s}$ for some $\rho_1 > 0$. Then*

(a) *There is $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ the origin is a hyperbolic equilibrium point of X_ε. Moreover in the coordinates introduced in the Hypotheses 1 and 2 we can represent the stable manifold locally as the graph*

$$\{x^s, y^s, \varphi_\varepsilon^1(x^s, y^s, z^s), z^s, \varphi_\varepsilon^2(x^s, y^s, z^s)\}$$

of an analytic map $(x^u, y^u, z^u) = \varphi_\varepsilon(x^s, y^s, z^s)$ defined on $P_{\rho_2} \times B_{\rho_2}$ for some $\rho_2 > 0$.

(b) *For $\varepsilon \searrow 0$, φ_ε converges to h uniformly on $P_{\rho_2} \times B_{\rho_2}$, more precisely*

$$\sup_{P_{\rho_2} \times B_{\rho_2}} |\varphi_\varepsilon - h| = O(\varepsilon). \tag{2.20}$$

Remark 2.4. *Here we only treat the stable manifold since results for the unstable manifold are immediately obtained by reversing the time.*

3. **Preliminary transformations and characterization of the limit function** h**.** Let us consider once again Equation (2.6). Its linear part can be simplified by using the following general result.

Lemma 3.1. *Let L_ε be a family of linear maps of a finite dimensional space E to itself of the form*

$$L_\varepsilon = A^0 + \varepsilon A^1 + O(\varepsilon^2),$$

where ε is a parameter close to 0*. Suppose that there is some splitting of the space $E = E_1 \oplus \cdots \oplus E_k$ for which we can write A^0 as a block diagonal matrix*

$$A^0 = \mathrm{diagonal}\,[S_1, \dots, S_k]$$

and that the spectra of the S_i are mutually disjoint.

Then there exists a linear change of variables of the form $\mathrm{Id} + \varepsilon Q$ *such that*

$$(\mathrm{Id} + \varepsilon Q)^{-1}.L_\varepsilon.(\mathrm{Id} + \varepsilon Q) = A^0 + \varepsilon B^1 + O(\varepsilon^2),$$

where B^1 is block diagonal (the blocks having the same dimensions as the blocks of A^0). More concretely, the non-zero blocks of B^1 are the diagonal blocks of A^1.

In the proof we will use the following result we quote from [11, Theorem 2, page 414].

Theorem 3.2. *The equation $AX + XB = C$ has a unique solution if and only if the matrices A and $-B$ have no eigenvalues in common.*

Proof of Lemma 3.1. We must determine Q such that

$$(A^0 + \varepsilon A^1).(\mathrm{Id} + \varepsilon Q) = (\mathrm{Id} + \varepsilon Q).(A^0 + \varepsilon B^1) + O(\varepsilon^2)$$

or equivalently

$$A^0.Q + A^1 = B^1 + Q.A^0 + O(\varepsilon).$$

Then it suffices to find Q such that

$$A^0.Q - Q.A^0 = B^1 - A^1. \tag{3.1}$$

Given a matrix C, we represent it in blocks according to the decomposition $E = E_1 \oplus \cdots \oplus E_k$, as $C = (C_{ij})$, where $C_{ij} : E_i \to E_j$ are defined by $C_{ij}x = \pi_j C_{|E_i} x$. We denote by $\delta_{ij} : E_i \to E_j$ the linear map defined by $\delta_{ii} = \mathrm{Id}$ and $\delta_{ij} = 0$ if $i \neq j$. Using the above introduced notation we can write $(A^0)_{ij} = \delta_{ij} S_j$. Hence a short calculation gives

$$(A^0.Q - Q.A^0)_{ij} = S_i.Q_{ij} - Q_{ij}.S_j.$$

Now if we take $B^1_{ii} = A^1_{ii}$ and $B^1_{ij} = 0$ if $i \neq j$ equation (3.1) is equivalent to

$$S_i Q_{ii} - Q_{ii} S_i = 0, \tag{3.2}$$

$$S_i Q_{ij} - Q_{ij} S_j = -A^1_{ij}. \tag{3.3}$$

We can solve (3.2) by choosing $Q_{ii} = 0$ (certainly not the unique solution). By the hypothesis on the spectra we can solve (3.3) by using Theorem 3.2. This finishes the proof. Q.E.D.

Lemma 3.3. *If $\beta_k \neq -\beta_l$ for every k, l there is a linear change of variables of the form $\mathrm{Id} + \varepsilon Q$, that transforms (2.6) into the form*

$$\left\{ \begin{array}{l} x' = iB.x + \varepsilon b_{11}.x \\ \bar{x}' = -iB.\bar{x} + \varepsilon \bar{b}_{11}.\bar{x} \\ z' = \varepsilon a_{33}.z \end{array} \right\rangle + \varepsilon O(|(x, \bar{x}, z)|^2) + O(\varepsilon^2). \tag{3.4}$$

If moreover $\beta_k \neq \beta_l$ for all $k \neq l$ we can achieve that b_{11} is a diagonal matrix, say:

$$b_{11} = \mathrm{diagonal}[b^1_{11}, \ldots, b^m_{11}]. \tag{3.5}$$

More concretely, b_{11} is the diagonal of $(a_{11} + a_{22} + i(a_{21} - a_{12}))/2$.

Proof. We have that

$$DX_\varepsilon(0,0,0) = \begin{bmatrix} iB & 0 & 0 \\ 0 & -iB & 0 \\ 0 & 0 & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ \bar{b}_{12} & \bar{b}_{11} & \bar{b}_{13} \\ b_{31} & \bar{b}_{31} & a_{33} \end{bmatrix} + O(\varepsilon^2).$$

The conditions on the betas imply that the matrices $iB, -iB$ and 0 have no common eigenvalues. Note that $\beta_k \neq -\beta_k$ implies $\beta_k \neq 0$. Then Lemma 3.1 gives the result. Under the condition $\beta_k \neq \beta_l$ for $k \neq l$ we can view the matrix diagonal$[iB, -iB, 0]$ as a matrix consisting of $2m + 1$ blocks, instead of 3, that is iB, $-iB$ consisting of m blocks each and then apply Lemma 3.1. Q.E.D.

Using the definition of R and the previous lemma we have

$$R(x, \bar{x}, z) = B^{11}.x + O(|(x, \bar{x}, z)|^2)$$

with B^{11} diagonal. We denote $Y_{0,\varepsilon}$ the vector field obtained by rewriting (3.4) using cylindrical coordinates (2.14) and truncating at order ε:

$$Y_{0,\varepsilon}:\left\{\begin{array}{lcll} r_j' &=& \varepsilon\,\mathrm{Re}(R^j(re^{i\theta},re^{-i\theta},z)e^{-i\theta_j}), & 1\le j\le m,\\ \theta_j' &=& \beta_j+\frac{\varepsilon}{r_j}\,\mathrm{Im}(R^j(re^{i\theta},re^{-i\theta},z)e^{-i\theta_j}), & 1\le j\le m,\\ z' &=& \varepsilon S(re^{i\theta},re^{-i\theta},z). & \end{array}\right. \tag{3.6}$$

Then still using cylindrical coordinates and the functions defined at (2.15) and (2.16) we can introduce a second auxiliary vector field:

$$Y^0:\left\{\begin{array}{l} r'=C_1(r,z),\\ z'=C_3(r,z). \end{array}\right. \tag{3.7}$$

Note that although r_j is a non-negative radius the vector field Y^0 extends analytically to $r_j<0$. The next lemma provides a symmetry property which will be used to get symmetries for the functions C_1 and C_3.

Lemma 3.4. *Let F be an analytic function defined in a neighbourhood of 0. Let us consider a function G of the form*

$$G(r)=\int_0^{2\pi}F(re^{i\varphi},re^{-i\varphi})\,d\varphi.$$

Then for r small $G(-r)=G(r)$.

Proof. The result follows from the straightforward calculation

$$\begin{aligned} G(-r) &= \int_0^{2\pi}F(-re^{i\varphi},-re^{-i\varphi})\,d\varphi\\ &= \int_0^{2\pi}F(re^{i(\varphi+\pi)},re^{-i(\varphi+\pi)})\,d\varphi\\ &= \int_\pi^{3\pi}F(re^{i\psi},re^{-i\psi})\,d\psi\\ &= \Big(\int_\pi^{2\pi}+\int_{2\pi}^{3\pi}\Big)F(re^{i\psi},re^{-i\psi})\,d\psi\\ &= \Big(\int_\pi^{2\pi}+\int_0^{\pi}\Big)F(re^{i\psi},re^{-i\psi})\,d\psi\\ &= G(r). \end{aligned}$$

Q.E.D.

As a consequence of this lemma and Fubini's theorem we have

Corollary 3.5. *The functions $r_kC_1^k(r_1,\ldots,r_m,z)$ and $C_3(r_1,\ldots,r_m,z)$ are even in each r_j.*

We are now able to prove the hyperbolicity of the origin for the vector fields Y^0 and Z^0.

Proposition 3.6. *1) The linear parts of Y^0 and Z^0 at the origin are*

$$DY^0(0,0)=\mathrm{diagonal}[\mathrm{Re}\,b_{11}^1,\ldots,\mathrm{Re}\,b_{11}^m,\ a_{33}^s,\ a_{33}^u], \tag{3.8}$$

$$DZ^0(0,0,0)=\mathrm{diagonal}[\mathrm{Re}\,b_{11}^1,\ldots,\mathrm{Re}\,b_{11}^{j_s},\ a_{33}^s,\ a_{33}^u]. \tag{3.9}$$

2) Under Hypotheses 1 and 2 the stable invariant manifolds of Y^0 and Z^0 can be locally represented by the graphs

$$\{r^s,0,z^s,h^s(r^s,z^s)\}, \tag{3.10}$$

and

$$\{r^s, z^s, h^s(r^s, z^s)\},$$

respectively for $\{|r_1| < \rho_1, \ldots, \ |r_{j_s}| < \rho_1, \ \|z^s\| < \rho_1\}$. *Moreover*

$$h^s(r_1, \ldots, r_{j_s}, z^s) = \tilde{h}^s(r_1^2, \ldots, r_{j_s}^2, z^s) \tag{3.11}$$

with $\tilde{h}^s$ *analytic.*

Note that while $\operatorname{Re} b_{11}^j$ are real numbers, a_{33}^s and a_{33}^u are matrices.

Proof We calculate the linear parts of Y^0 and Z^0 at the origin. For $1 \le k \le m$ we have

$$\begin{aligned} C_1^k(r, z) &= \frac{1}{(2\pi)^m} \int_{\mathbb{T}^m} \Big(\operatorname{Re} b_{11}^k \, r_k e^{i\theta_k} + O(|(re^{i\theta}, re^{-i\theta}, z)|^2) \Big) e^{-i\theta_k} \, d\theta \\ &= \operatorname{Re} b_{11}^k r_k + O(|(r, z)|^2) \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} C_3(r, z) &= \frac{1}{(2\pi)^m} \int_{\mathbb{T}^m} \Big(a_{33}.z + O(|(re^{i\theta}, re^{-i\theta}, z)|^2) \Big) \, d\theta \\ &= a_{33}.z + O(|(r, z)|^2). \end{aligned} \tag{3.13}$$

This proves (3.8) and (3.9).

Then, as a consequence of Hypotheses 1 and 2 it is well known that Y^0 and Z^0 have stable and unstable invariant manifolds, which locally are graphs of analytic maps.

We look for the stable manifold as a graph of $(r^u, z^u) = h(r^s, z^s)$ with $h = (h_1, h_2)$. We write the invariant condition for the graph of h:

$$\begin{aligned} (r^u)' &= D_1 h_1 \ (r^s)' + D_2 h_1 \ (z^s)', \\ (z^u)' &= D_1 h_2 \ (r^s)' + D_2 h_2 \ (z^s)', \end{aligned}$$

which gives

$$C_1^u = D_1 h_1 \ C_1^s + D_2 h_1 \ C_3^s, \tag{3.14}$$

$$C_3^u = D_1 h_2 \ C_1^s + D_2 h_2 \ C_3^s, \tag{3.15}$$

where $C_{1,3}^{s,u}$ are evaluated on $(r^s, h_1(r^s, z^s), z^s, h_2(r^s, z^s))$. Note that $h_1 = 0$ is a solution of (3.14) because $C_1(r^s, 0, z^s, z^u) = 0$ by Corollary 3.5. Then (3.15) becomes

$$C_3^u(r^s, 0, z^s, h_2) = D_2 h_2(r^s, z^s) \ C_3^s(r^s, 0, z^s, h_2)$$

which is the condition for invariance relative to the auxiliary system

$$\tilde{Y}^0 : \begin{cases} (r^s)' = C_1^s(r^s, 0, z), \\ (z^s)' = C_3^s(r^s, 0, z), \\ (z^u)' = C_3^u(r^s, 0, z). \end{cases} \tag{3.16}$$

Then $h_1 = 0$. We denote $h^s = h_2$. Let $\mathcal{R}_k(r) = (r_1, \ldots, -r_k, \ldots, r_m)$. The symmetry properties given by Corollary 3.5 imply that if $(r(t), z(t))$ is a solution of (3.7) then $(\mathcal{R}_k r(t), z(t))$ is also a solution. This implies that if $(r_0, z_0) \in W^s$ then $(\mathcal{R}_k r_0, z_0) \in W^s$. Applying this last property to $\{r^s, 0, z^s, h^s(r^s, z^s)\}$ we obtain that $\{\mathcal{R}_k r^s, 0, z^s, h^s(r^s, z^s)\} \in W^s$ and hence $h^s(r^s, z^s) = h^s(\mathcal{R}_k r^s, z^s)$. Since $k \in \{1, \ldots, m\}$ and h^s is analytic (3.11) follows. Q.E.D.

4. **Normal form.** The next step is to introduce and use a suitable normal form. We first describe the procedure in an informal and somewhat imprecise way and then we apply it rigorously to find the normal form we are interested in. Let X_ε be a family of vector fields of the form

$$X_\varepsilon(v) = A.v + \varepsilon f(v) + O(\varepsilon^2)\,. \tag{4.1}$$

In the following we will use it for X_ε as in (2.1), or rather, in more suitable coordinates, as in (3.4); in this latter case $A = \text{diagonal}[iB, -iB, 0]$ and $v = (x, \bar{x}, z)$. We want to simplify X_ε based on the properties of A.

Let E be some function space of C^1 functions, $u(v)$, defined on some domain V. We define the linear operator ad_A by the formula

$$(ad_A(u))\,(v) := Du(v).A.v - A.u(v). \tag{4.2}$$

Let F be some function space containing f. One can think of E, F being spaces of convergent power series on some polydisk. We denote H the image of ad_A in F and by G some complementary function space of H in F, i.e. $F = H \oplus G$. Then we can write

$$f = f_H + f_G \tag{4.3}$$

with $f_H \in H$ and $f_G \in G$. Hence there exists $u \in E$ such that

$$ad_A(u) = f_H. \tag{4.4}$$

Proposition 4.1. *The previous definitions and formulas (4.2) and (4.3) imply that if u satisfies* (4.4) *the change of variables $w = \psi_\varepsilon(v) := v - \varepsilon u(v)$ conjugates X_ε in (4.1) to*

$$(\psi_\varepsilon)_* X_\varepsilon(w) = A.w + \varepsilon f_G(w) + O(\varepsilon^2). \tag{4.5}$$

Proof. It is clear that $\psi_\varepsilon^{-1}(w) = w + \varepsilon u(w) + O(\varepsilon^2)$, then a straightforward computation gives:

$$\begin{aligned}
(\psi_\varepsilon)_* X_\varepsilon(w) &= D\psi_\varepsilon(v).X_\varepsilon(v) \\
&= (\mathrm{Id} - \varepsilon Du(v)).(A.v + \varepsilon f(v) + O(\varepsilon^2)) \\
&= A.v + \varepsilon f(v) - \varepsilon Du(v).A.v + O(\varepsilon^2) \\
&= A.(w + \varepsilon u(w)) + \varepsilon f(w) - \varepsilon Du(w).A.w + O(\varepsilon^2) \\
&= A.w - \varepsilon(ad_A(u))(w) + \varepsilon f(w) + O(\varepsilon^2),
\end{aligned}$$

where $v = \psi_\varepsilon^{-1}(w)$. Choosing u in the definition of ψ_ε verifying (4.4) and using the decomposition of f in (4.3) we get

$$(\psi_\varepsilon)_* X_\varepsilon(w) = A.w + \varepsilon f_G(w) + O(\varepsilon^2). \tag{4.6}$$

Q.E.D.

Proposition 4.2. *Let Z_ε be a vector field of the form* (2.6) *whose right-hand side is assumed to be analytic with respect to $x, \bar{x}, z$ on $P_\rho \times B_\rho$, hence it has the following expansion for R and S:*

$$R(x, \bar{x}, z) = \sum_{k=1}^{m} \sum_{p,q \in \mathbb{N}^m} R^k_{p,q}(z) x^p \bar{x}^q e_k, \tag{4.7}$$

$$S(x, \bar{x}, z) = \sum_{p,q \in \mathbb{N}^m} S_{p,q}(z) x^p \bar{x}^q, \tag{4.8}$$

where $\{e_1, \dots, e_m\}$ *denotes the canonical vector space basis of* $\mathbb{C}^m$. *Assume that* $\beta_1, \dots, \beta_m$ *satisfy Hypothesis 3, then there exists a near to the identity change of variables of the form*

$$(x, \bar{x}, z) \mapsto \psi_\varepsilon(x, \bar{x}, z) = (x, \bar{x}, z) - \varepsilon u(x, \bar{x}, z), \tag{4.9}$$

with an analytic function u, *such that in the new variables (which we still denote by* $(x, \bar{x}, z)$*) the vector field* Z_ε *takes the form* $Z_{\varepsilon,N} = (\psi_\varepsilon)_* Z_\varepsilon$:

$$\begin{cases} x' &= iB.x + \varepsilon \sum_{|q|=1}^{\infty} |x_1|^{2q_1} \dots |x_m|^{2q_m} \sum_{k=1}^{m} R^k_{q+e_k,q}(z) x^{e_k} e_k + O(\varepsilon^2)\,, \\ \bar{x}' &= \text{ complex conjugate of the first component}\,, \\ z' &= \varepsilon \sum_{|q|=1}^{\infty} S_{q,q}(z) |x_1|^{2q_1} \dots |x_m|^{2q_m} + O(\varepsilon^2)\,, \end{cases} \tag{4.10}$$

where the right-hand side is analytic in $P_{\rho'} \times B_\rho$ *with* $0 < \rho' < \rho$.

Proof. To simplify the $\partial/\partial x$ and $\partial/\partial \bar{x}$ components of the vector field we look for u of the following form

$$u(x, \bar{x}, z) = \Big(\sum_{k=1}^{m} \sum_{p,q \in \mathbb{N}^m} u^k_{p,q}(z) x^p \bar{x}^q e_k,\ \text{complex conjugate of the first component}\ , \sum_{k=1}^{m} \sum_{p,q \in \mathbb{N}^m} \tilde{u}^k_{p,q}(z) x^p \bar{x}^q e_k \Big). \tag{4.11}$$

One can explicitly compute the action of ad_A from its definition (4.2) and the given form for $A = \text{diagonal}[iB, -iB, 0]$, to obtain:

$$\begin{aligned} (ad_A(u))\,(x, \bar{x}, z) \quad = \quad & \Big(\sum_{k=1}^{m} \sum_{p,q} i\langle \beta, p - q - e_k \rangle u^k_{p,q}(z) x^p \bar{x}^q e_k, \\ & \text{complex conjugate of the first component}\ , \\ & \sum_{k=1}^{m} \sum_{p,q} i\langle \beta, p - q \rangle \tilde{u}^k_{p,q}(z) x^p \bar{x}^q e_k \Big). \end{aligned} \tag{4.12}$$

By Hypothesis 3 we have that $\langle \beta, p - q - e_k \rangle = 0 \ \Leftrightarrow\ p = q + e_k$ and $\langle \beta, p - q \rangle = 0 \ \Leftrightarrow\ p = q$. Hence the image of ad_A contains, at formal level, series of the form

$$\Big(\sum_{k=1}^{m} \sum_{p \neq q + e_k} R^k_{p,q}(z) x^p \bar{x}^q e_k, \text{conjugate of the first component}\ , \sum_{k=1}^{n} \sum_{p \neq q} S^k_{p,q}(z) x^p \bar{x}^q e_k \Big). \tag{4.13}$$

Let us then decompose R and S in (4.7) and (4.8) according to the splitting introduced in (4.3), as:

$$R = R_H + R_G, \qquad S = S_H + S_G,$$

where the functions in the decomposition are explicitly given by:

$$R_H = \sum_{k=1}^{m} \sum_{p \neq q + e_k}, \qquad R_G = \sum_{k=1}^{m} \sum_{p = q + e_k}, \qquad S_H = \sum_{k=1}^{n} \sum_{p \neq q}, \qquad S_G = \sum_{k=1}^{n} \sum_{p = q}.$$

By the analyticity hypothesis of the involved vector fields, there are positive constants M and ρ such that

$$|R^k_{p,q}(z)| \leq M \rho^{|p|+|q|}, \qquad |S^k_{p,q}(z)| \leq M \rho^{|p|+|q|} \tag{4.14}$$

for all k and uniformly in z.

Thus to conclude the proof, i.e. to transform (2.2) into (4.10), it is enough to find a convergent u of the form (4.11) such that $ad_A(u) = (R_H, \bar{R}_H, S_H)$. Because of (4.12) we can take

$$\begin{aligned} u(x,\bar{x},z) \;&=\; \Big(\sum_{k=1}^{m}\sum_{p\neq q+e_k}\frac{R^k_{p,q}(z)}{i\langle\beta,p-q-e_k\rangle}x^p\bar{x}^q e_k, \\ &\quad \text{complex conjugate of the first component },\\ &\quad \sum_{k=1}^{n}\sum_{p\neq q}\frac{S^k_{p,q}(z)}{i\langle\beta,p-q\rangle}x^p\bar{x}^q e_k\Big). \end{aligned}$$

Using the arithmetical condition in Hypothesis 3, given $\varepsilon > 0$ we can estimate

$$\Big|\frac{1}{\langle\beta,p-q-e_k\rangle}\Big| \le Ce^{(\sigma+\varepsilon)|p-q-e_k|} \le Ce^{(\sigma+\varepsilon)}e^{(\sigma+\varepsilon)(|p|+|q|)} \tag{4.15}$$

for some $C > 0$ and hence

$$\Big|\frac{R^k_{p,q}(z)}{i\langle\beta,p-q-e_k\rangle}\Big|^{\frac{1}{|p+q|}} \le (Ce^{(\sigma+\varepsilon)}M)^{\frac{1}{|p+q|}}e^{(\sigma+\varepsilon)}\rho. \tag{4.16}$$

Similarly

$$\Big|\frac{S^k_{p,q}(z)}{i\langle\beta,p-q\rangle}\Big|^{\frac{1}{|p+q|}} \le (Ce^{(\sigma+\varepsilon)}M)^{\frac{1}{|p+q|}}e^{(\sigma+\varepsilon)}\rho.$$

Therefore we get convergence on the polydisk $P_{\rho'}$ with $\rho' = \rho/e^{(\sigma+\varepsilon)}$ uniformly in $z \in B_\rho$. Q.E.D.

Proposition 4.3. *Let $Z_{\varepsilon,N,T}$ be the vector field obtained from $Z_{\varepsilon,N}$ in (4.10) dropping the $O(\varepsilon^2)$ terms, then in cylindrical coordinates this vector field yields the equations:*

$$\begin{cases} r' &= \varepsilon C_1(r,z)\\ z' &= \varepsilon C_3(r,z)\,, \end{cases} \tag{4.17}$$

i.e. up to a factor ε it is the vector field Y^0 defined previously in (3.7).

Proof. Using the expansions (4.7) and (4.8) and the formulas in (2.15) and (2.16) we have

$$C^k_1(r_1,\dots,r_m,z) = \frac{1}{(2\pi)^m}\int_{\mathbb{T}^m}\mathrm{Re}\big(\sum_{p,q\in\mathbb{N}^m}R^k_{p,q}(z)x^p\bar{x}^q e^{-i\theta_k}\big)\,d\theta\,, \qquad 1\le k\le m\,. \tag{4.18}$$

We split up the summation in (4.18) as $\sum_{p,q\in\mathbb{N}^m} = \sum_{p=q+e_k} + \sum_{p\neq q+e_k}$. Then we observe that all the terms involving $p \neq q+e_k$ do not contribute to (4.18) because of:

$$\begin{aligned} \int_{\mathbb{T}^m} x^p\bar{x}^q e^{-i\theta_k}\,d\theta \;&=\; \int_{\mathbb{T}^m} r_1^{p_1+q_1}\dots r_m^{p_m+q_m}e^{i\theta_1(p_1-q_1)}\dots e^{i\theta_k(p_k-q_k-1)}\dots e^{i\theta_m(p_m-q_m)}\,d\theta\\ &=\; 0 \end{aligned} \tag{4.19}$$

by Fubini's theorem. While the terms with $p = q+e_k$ give:

$$\begin{aligned} \int_{\mathbb{T}^m} x^{q+e_k}\bar{x}^q e^{-i\theta_k}\,d\theta \;&=\; \int_{\mathbb{T}^m} r_1^{2q_1}\dots r_k^{2q_k+1}\dots r_m^{2q_m}e^0 d\theta\\ &=\; (2\pi)^m r_1^{2q_1}\dots r_k^{2q_k+1}\dots r_m^{2q_m}. \end{aligned} \tag{4.20}$$

We can thus conclude that:

$$C_1^k(r_1,\dots,r_m,z)=\sum_{q\in\mathbb{N}^m}\mathrm{Re}(R^k_{q+e_k,q}(z)r_1^{2q_1}\dots r_k^{2q_k+1}\dots r_m^{2q_m}). \tag{4.21}$$

On the other hand, expressing $Z_{\varepsilon,N,T}$ in cylindrical coordinates we can use a formula similar to (3.6) and get

$$r_k'=\varepsilon\,\mathrm{Re}(r_1^{2q_1}\dots r_m^{2q_m}R^k_{q+e_k,q_k}(z)r_ke^{i\theta_k}.e^{-i\theta_k}). \tag{4.22}$$

Comparing the right-hand sides of (4.21) and (4.22) we see that they are equal up to a factor ε. A completely analogous calculation can be performed for C_3 and this concludes the proof. Q.E.D.

Because of this Proposition 4.3 and since the equations in (4.17) are independent of the angles $\theta_1,\dots,\theta_m$ by Proposition 3.6 we have

Proposition 4.4. *The graph defined by (2.19) is invariant for $Z_{\varepsilon,N,T}$ for any $\varepsilon>0$.*

In order to finish the proof of Theorem 2.3 we repeat the arguments in [4, Theorem 4, page 141] concerning the comparison of the time one maps of $Z_{\varepsilon,N,T}$ and $Z_{\varepsilon,N}$, which differ by $O(\varepsilon^2)$ and then we go back to the original variables.

5. **Setting and main result for diffeomorphisms.** We consider families of diffeomorphisms $F_\varepsilon:U\subset\mathbb{R}^{2m}\times\mathbb{R}^n\to\mathbb{R}^{2m}\times\mathbb{R}^n$ close to a product of rotations having the form

$$F_\varepsilon(v)=A.v+\varepsilon f_\varepsilon(v) \tag{5.1}$$

with $A=\mathrm{diagonal}[G_1,\dots,G_m,\mathrm{Id}_n]$, where G_j are rotations in $\mathbb{R}^2$ of angle $\beta_j\neq 0$ and $f_\varepsilon(0)=0$.

Using the complex variables (2.4) and (2.5) we can write F_ε as

$$F_\varepsilon(x,\bar x,z)=\begin{pmatrix} C.x+\varepsilon M(x,\bar x,z)\\ \bar C.\bar x+\varepsilon\bar M(x,\bar x,z)\\ z+\varepsilon N(x,\bar x,z)\end{pmatrix}+O(\varepsilon^2), \tag{5.2}$$

with $C=\mathrm{diagonal}[e^{i\beta_1},\dots,e^{i\beta_m}]$. Let us introduce the block matrix $(b_{ij})_{1\le i,j\le 3}=D(M,\bar M,N)(0,0,0)$ and let us denote the diagonal elements of b_{11} by $b_{11}^1,\dots,b_{11}^m$.

In cylindrical coordinates the diffeomorphism can be rewritten as $F_\varepsilon=(F_{1,\varepsilon},F_{2,\varepsilon},F_{3,\varepsilon})$ with

$$\begin{aligned} F^k_{1,\varepsilon}(r,\theta,z)&=r_k+\varepsilon\,\mathrm{Re}[M^k(re^{i\theta},re^{-i\theta},z)e^{-i(\beta_k+\theta_k)}]+O(\varepsilon^2),\\ F^k_{2,\varepsilon}(r,\theta,z)&=\theta_k+\beta_k+(\varepsilon/r_k)\,\mathrm{Im}[M^k(re^{i\theta},re^{-i\theta},z)e^{-i(\beta_k+\theta_k)}]+O(\varepsilon^2),\\ F_{3,\varepsilon}(r,\theta,z)&=z+\varepsilon N(re^{i\theta},re^{-i\theta},z)+O(\varepsilon^2).\end{aligned} \tag{5.3}$$

We consider the following averaged functions

$$E_1^k(r,z)=\frac{1}{(2\pi)^m}\int_{\mathbb{T}^m}\mathrm{Re}[M^k(re^{i\theta},re^{-i\theta},z)e^{-i(\beta_k+\theta_k)}]\,d\theta, \tag{5.4}$$

$E_1=(E_1^1,\dots,E_1^m)$ and

$$E_3(r,z)=\frac{1}{(2\pi)^m}\int_{\mathbb{T}^m}N(re^{i\theta},re^{-i\theta},z)\,d\theta. \tag{5.5}$$

As a consequence of Lemma 3.4 the functions $r_kE_1^k(r,z)$ and $E_3(r,z)$ are even with respect to each r_j. Let Y^0 be the auxiliary equation

$$Y^0:\begin{cases} r'=E_1(r,z),\\ z'=E_3(r,z).\end{cases} \tag{5.6}$$

In the same way as in Proposition 3.6 we obtain that

$$DY^0(0,0) = \text{diagonal}[\text{Re}(b_{11}^1 e^{-i\beta_1}), \ldots, \text{Re}(b_{11}^m e^{-i\beta_m}), b_{33}].$$

We introduce two hypotheses related to hyperbolicity analogous to Hypotheses 1 and 2 of Section 2.

Hypothesis 4. *The real parts of $b_{11}^j e^{-i\beta_j}$, $j = 1, \ldots, m$, are different from zero. It is not restrictive to assume that the real parts are ordered as*

$$\text{Re}(b_{11}^1 e^{-i\beta_1}) \le \cdots \le \text{Re}(b_{11}^{j_s} e^{-i\beta_{j_s}}) < 0 < \text{Re}(b_{11}^{j_s+1} e^{-i\beta_{j_s+1}}) \le \cdots \le \text{Re}(b_{11}^m e^{-i\beta_m}) \tag{5.7}$$

with $0 \le j_s \le m$. We write $x = (x^s, x^u)$ with $x^s = (x_1, \ldots, x_{j_s})$ and $x^u = (x_{j_s+1}, \ldots, x_m)$.

Hypothesis 5. *The matrix b_{33} is hyperbolic in the sense that all its eigenvalues have real part different from zero. Hence, up to a linear change of variables only in $z \in \mathbb{R}^n$, we will assume that there is a decomposition of $\mathbb{R}^n$ such that, with respect to it, b_{33} has the form*

$$b_{33} = \begin{bmatrix} b_{33}^s & 0 \\ 0 & b_{33}^u \end{bmatrix}, \tag{5.8}$$

where the eigenvalues of b_{33}^s, resp. b_{33}^u, have negative, resp. positive real part. We write the variable $z \in \mathbb{R}^n$ with respect to this decomposition as $z = (z^s, z^u)$.

Also we introduce an arithmetical condition on the betas.

Hypothesis 6. *Let $\beta = (\beta_1, \ldots, \beta_m)$. For $\vec{n} \in \mathbb{Z}^m$ and $k \in \mathbb{Z}$ we have*

$$e^{i\langle \beta, \vec{n}\rangle} = 1 \quad \Longleftrightarrow \quad |\vec{n}| = 0.$$

Moreover

$$\sigma := \overline{\lim_{|\vec{n}| \to +\infty}} \frac{1}{|\vec{n}|} \log |e^{i\langle \beta, \vec{n}\rangle} - 1|^{-1} < \infty. \tag{5.9}$$

Remark 5.1. *If β satisfies*

$$|\langle \beta, \vec{n}\rangle - 2\pi\ell| \ge K|\vec{n}|^{-\tau} \quad \text{for all} \quad \vec{n} \in \mathbb{Z}^m \setminus \{0\}, \ell \in \mathbb{Z} \tag{5.10}$$

for some $K > 0$ and $\tau > 0$ then (5.9) *holds with $\sigma = 0$.*

Since the set of betas satisfying (5.10) *with $\tau > m$ has full Lebesgue measure* [12] *then the set of betas satisfying* (5.9) *also has full Lebesgue measure in $\mathbb{R}^m$.*

In the same way as in Proposition 3.6 we have

Proposition 5.2. *Under Hypotheses 4 and 5 the origin is a fixed equilibrium point of* (5.6) *and moreover its local stable manifold can be represented as the graph*

$$\{r^s, 0, z^s, h^s(r^s, z^s)\}, \tag{5.11}$$

where $h^s(r_1, \ldots, r_{j_s}, z^s) = \tilde{h}^s(r_1^2, \ldots, r_{j_s}^2, z^s)$ with $\tilde{h}^s$ analytic.

Theorem 5.3. *Let F_ε be a family of diffeomorphisms as in (5.1) and suppose that the functions f_ε are analytic on $P_{\rho_0} \times B_{\rho_0} \subset \mathbb{C}^{2m} \times \mathbb{C}^n$. Suppose that Hypotheses 4, 5, and 6 are satisfied. Consider the following local graph of the analytic function $(x^u, y^u, z^u) = h(x^s, y^s, z^s)$ defined by*

$$\{x^s, y^s, 0, 0, z^s, h^s(|x_1 + iy_1|, \ldots, |x_{j_s} + iy_{j_s}|, z^s)\}, \tag{5.12}$$

defined on $P_{\rho_1} \times B_{\rho_1}$, where h^s is the function introduced in (5.11).

Then
(a) There is $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ the origin is a hyperbolic fixed point of F_ε. Moreover in the coordinates introduced in Hypotheses 4 and 5 we can represent the stable manifold locally as the graph

$$\{x^s, y^s, \varphi^1_\varepsilon(x^s, y^s, z^s), z^s, \varphi^2_\varepsilon(x^s, y^s, z^s)\}.$$

of an analytic map $(x^u, z^u) = \varphi_\varepsilon(x^s, z^s)$ defined on $P_{\rho_2} \times B_{\rho_2} \subset \mathbb{C}^{2j_s} \times \mathbb{C}^{k_s}$ for some ρ_2.
(b) For $\varepsilon \searrow 0$, φ_ε converges to h uniformly on $P_{\rho_2} \times B_{\rho_2}$.

Remark 5.4. *Hypothesis 6 is essential as the counterexample in* [4] *shows.*

The analogous result for the unstable manifold is readily obtained by considering F_ε^{-1}.

The proof goes along the same lines as the one of Theorem 5 in [4]. The main difference is the proof of the analogous of Proposition 3 in that reference which we now state and prove. It deals with the embedding of the family F_ε into a flow up to order ε^2.

Proposition 5.5. *Let F_ε be the family of diffeomorphisms given by* (5.2) *with M, N analytic. Assume that Hypothesis 6 holds. Then there is a vector field of the form*

$$X_\varepsilon(v) = L.v + \varepsilon g(v)$$

such that the time one map H_ε of X_ε satisfies the condition

$$|F_\varepsilon(v) - H_\varepsilon(v)| = O(\varepsilon^2)$$

uniformly in z on a set of the form $P_{\rho_0} \times B_{\rho_0} \subset \mathbb{C}^{2m} \times \mathbb{C}^n$.

Proof We look for a vector field of the form $X_\varepsilon(v) = L.v + \varepsilon g(v)$, $v = (x, \bar{x}, z)$, with $L = \text{diagonal}[iB, -iB, 0]$ and $B = \text{diagonal}[\beta_1, \dots, \beta_m]$. Note that $e^L = \text{diagonal}[C, \bar{C}, \text{Id}]$. Let $\phi_\varepsilon(t, v)$ denote the flow of X_ε. We develop it with respect to ε

$$\phi_\varepsilon(t, v) = \phi_0(t, v) + \varepsilon \frac{\partial \phi_\varepsilon}{\partial \varepsilon}(t, v)|_{\varepsilon=0} + O(\varepsilon^2). \tag{5.13}$$

Clearly $\phi_0(t, .) = e^{Lt}$. The derivative $\psi(t, v) = \dfrac{\partial \phi_\varepsilon}{\partial \varepsilon}(t, v)|_{\varepsilon=0}$ satisfies the variational equation with respect to ε

$$\frac{\partial \psi}{\partial t}(t, v) = L.\psi(t, v) + g(e^{At}.v), \qquad \psi(0, v) = 0. \tag{5.14}$$

The variation of constants formula gives:

$$\psi(t, v) = \int_0^t \exp(L(t-s)).g(e^{As}.v)\, ds. \tag{5.15}$$

From (5.13) and (5.15) we have to solve

$$\psi(1, v) = \int_0^1 \exp(L(1-s)).g(e^{As}.v)\, ds = (M(v), \bar{M}(v), N(v)). \tag{5.16}$$

We expand M and N in $x, \bar{x}$ as

$$M(x, \bar{x}, z) = \sum_{p,q \in \mathbb{N}^m} c^1_{p,q}(z) x^p \bar{x}^q, \qquad N(x, \bar{x}, z) = \sum_{p,q \in \mathbb{N}^m} c^3_{p,q}(z) x^p \bar{x}^q, \tag{5.17}$$

where the sum converges on $P_{\rho_0} \times B_{\rho_0}$ for some ρ_0. We look for $g = (g^1, g^2, g^3)$, with $g^2 = \overline{g^1}$, of the form

$$g^j(x, \bar{x}, z) = \sum_{p,q \in \mathbb{N}^m} d^j_{p,q}(z) x^p \bar{x}^q.$$

Now we must proceed component-wise. For that we write $d^1_{p,q} = \sum_{k=1}^m d^{1,k}_{p,q} e_k$ and $c^1_{p,q} = \sum_{k=1}^m c^{1,k}_{p,q} e_k$, where $\{e_k\}_{1 \le k \le m}$ is the canonical basis of $\mathbb{R}^m$. With these notations we express the first component of (5.16) as

$$\int_0^1 \sum_{k=1}^m e^{i\beta_k(1-s)} \sum_{p,q} d^{1,k}_{p,q}(z) e_k e^{i\langle \beta, p-q \rangle s} x^p \bar{x}^q \, ds = \sum_{k=1}^m \sum_{p,q} c^{1,k}_{p,q}(z) e_k x^p \bar{x}^q.$$

Hence identifying coefficients, for $1 \le k \le m$ and $p, q \in \mathbb{N}^m$ we get

$$\int_0^1 e^{i\beta_k(1-s)} d^{1,k}_{p,q}(z) e^{i\langle \beta, p-q \rangle s} \, ds = c^{1,k}_{p,q}(z).$$

Evaluating the integral, after a short calculation we have

$$\begin{aligned} d^{1,k}_{p,q}(z) &= e^{-i\beta_k} \frac{i\langle \beta, p-q-e_k \rangle}{e^{i\langle \beta, p-q-e_k \rangle} - 1} c^{1,k}_{p,q}(z) &\quad& \text{if} \quad p-q-e_k \ne 0, \\ d^{1,k}_{p,q}(z) &= e^{-i\beta_k} c^{1,k}_{p,q}(z) && \text{if} \quad p-q-e_k = 0. \end{aligned}$$

In a completely similar way we find that

$$\begin{aligned} d^3_{p,q}(z) &= \frac{i\langle \beta, p-q \rangle}{e^{i\langle \beta, p-q \rangle} - 1} c^3_{p,q}(z) &\quad& \text{if} \quad p-q \ne 0, \\ d^3_{p,q}(z) &= c^3_{p,q}(z) && \text{if} \quad p-q = 0. \end{aligned}$$

Since we assume the series for f^j_0 to be convergent on a polydisk, there exist constants $C, \rho > 0$ such that

$$|c^j_{p,q}(z)| \le C\rho^{|p|+|q|}. \tag{5.18}$$

For $p - q - e_k \ne 0$, by Hypothesis 6 there exist $C_1 > 0$ and $\rho > 0$ such that

$$\begin{aligned} |d^{1,k}_{p,q}(z)| &\le |\beta|.|p-q-e_k||e^{i\langle \beta, p-q-e_k \rangle} - 1|^{-1} |c^{1,k}_{p,q}(z)| \\ &\le |\beta|(|p|+|q|+1)|e^{i\langle \beta, p-q-e_k \rangle} - 1|^{-1} C\rho^{|p|+|q|} \\ &\le |\beta|(|p|+|q|+1) C_1 e^{(\sigma+\varepsilon)|p-q-e_k|} \rho^{|p|+|q|} \\ &\le |\beta|(|p|+|q|+1) C_1 e^{(\sigma+\varepsilon)(|p|+|q|)} e^{\sigma+\varepsilon} C\rho^{|p|+|q|}. \end{aligned}$$

For $p - q = e_k$ we obviously have $|d^{1,k}_{p,q}(z)| = |c^{1,k}_{p,q}(z)|$. It is clear that the $(|p|+|q|)$-th root of $|d^{1,k}_{p,q}(z)|$ is bounded, implying convergence on $P_{\rho'} \times B_\rho$ with $\rho' < \rho / e^{\sigma+\varepsilon}$.

The estimates for $d^3_{p,q}$ are completely similar. Q.E.D.

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